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Research Article

Wavelet Leader and Multifractal Detrended Fluctuation Analysis of Market Efficiency: Evidence from the WAEMU Market Index

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Efficient markets hypothesis (EMH) has been a hot topic since its introduction in the 1960s. This problematic is a current topic and has been the subject of many studies with different methods. This paper examines the weak-form efficiency of the WAEMU stock exchange from 11/04/2008 to 23/08/2016. We combined the wavelets approaches and multifractal detrended fluctuation analysis (MF-DFA) to analyse the efficient market hypothesis of the BRVM10 index of the WAEMU regional stock change. Our findings show that the log return of BRVM10 index exhibits a persistent and multifractal process.

JEL codes: C02, C12, C40, G14

Keywords: Efficient market hypothesis; Wavelet; MF-DFA; BVRM10; Hurst exponent; Wavelet Leaders

1 Introduction

The many theoretical and empirical literature analyses the behaviour of the stock price reveals that the stock price is not only a value observed at the moment but contains all the information available on the market. This concept makes use of the efficient market hypothesis notion. Since the late 1950s, most of the works have been consulted as part of the financial markets efficient assumption. Thanks to this, finance has made great progress. This is a subject that has led to the production of rich and diverse works of great researchers. It has given rise to a very important revision with different dimensions, according to Kabbaj (2008), one can distinguish the allocative efficiency, the operational efficiency, and the informational efficiency.[1]

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[1] Allocative efficiency is to allocate funds to the most productive jobs; Pareto (1909) states that a market is deemed to be optimal Pareto if it is possible to increase the welfare of an agent without injury to one or more other agents. Operational efficiency refers to the organisation and the market structure. "The institutional, regulatory and technological characteristics, summarised under the name "microstructure of the markets", significantly in influence the securities supply and demand strategies of the various parties involved in the exchange process and, consequently, the prices training of assets listed" (Gillet & Szafard, 2004, 13).

The informationally efficient market hypothesis is derived from the work of Cootner (1964). Despite previous work, the paternity of this theory is generally attributed to Fama thanks to his thesis defended in 1965; he published an article in the Journal of Finance entitled "Efficient Capital Markets". Fama (1965) defines an efficient market like the one on which prices reflect the available information at all times. Always in the same direction, Fama (1970) classifies efficiency in three forms according to an information set. These include the weak, semi-strong and strong forms. The study of the behaviour of share prices is of great importance from the point of view of investors and policy-makers. Moreover, any extraordinary or random movement in prices that is grossly out of line with economic fundamentals raises concerns for both market practitioners and policy-makers alike. As a result, the understanding and analysis of stock price behaviour have interested academics in general and modellers in particular.

The efficient market hypothesis has certainly made enormous progress with academics and professionals thanks to the different approaches proposed. Only, the majority of studies carried out concern the developed stock markets. With the advent of specific phenomena such as globalisation or financial integration, the gaze is increasingly focused on the emerging or underdeveloped markets (Feldstein, 2000; Stulz, 1999; Stiglitz, 2005; Claessens et al., 2001; Chinn et al., 2015). However, it must be noted that the frequency of specific events such as the shocks or crisis (stock market crash (1929), the crash of October 1987 and the second half of 1997, the bubble 2000, the subprime crisis 2008). These facts revive debates on informational efficiency (Urrutia, 1995; Mignon & Abraham-Frois, 1998; Colmant et al., 2003; Gillet & Szafard, 2004; Lardic & Mignon, 2006; Khamis et al., 2018). One can wonder even if the repetitive crashes are not related to a problem of transparency or information used in the financial markets? Thus, the question of clarity on the financial markets and its utilisation is today at the core of debates between economists. In other words, the problems associated with the information disseminated at the stock exchange is a current topic that provokes discussion. Therefore, the questions revolve more around these reports to know if these repetitive crises cannot be factors of questioning the efficient market hypothesis?

This issue preoccupied today to all of the financial market participants. It is in this sense that we have access to our study on informational efficiency. This article focuses on the weak-form efficiency of the BRVM10 index of the WAEMU regional stock exchange. Thus, we are particularly interested in the predictability of stock prices in this market. There exist a battery of tests to examine the informationally efficient markets. However, it should be noted that ordinary Brownian movements are widely used in finance to study the evolution of the returns. They suppose that returns are continuous functions of time and follow the Gaussian distribution and that their increments are independent and stationary. However, studies have shown that approaching financial series with ordinary Brownian movements is not correct (Peters, 1991; Lo, 1997; Adam, 2001).

In the 1960s, Mandelbrot was the first to use this approach and showed the presence of scale laws in the markets. It highlights scale invariance (that is, a fractal market structure), which then allowed a market to be observed at any arbitrary scale as there was no more than free time to capture the fundamental behavioural structure of its fluctuations. All time scales were, therefore, suitable for statistical analysis. The fractal hypothesis of Mandelbrot drew attention to this problem and induced a new take into account of the scales. However, there is the same law of probability can model no particular reason for the variations corresponding to the short time horizon of the trader and those corresponding to the extensive background of the portfolio manager. This simultaneous coexistence of specific moments requires the instrumentation of analyses and multi-scale markets (time of traders, time of leaders, time of insurers, etc.) not to lose information on the phenomenon studied. The debate around the modelling of fractal markets is precisely on this point and involves the laws of dynamic scales (Levy, 1925, 1937; Mandelbrot, 1963; Fama, 1965; Oświe¸cimka et al., 2005; Wang et al., 2009; Yuan et al., 2009). Peters (1994) defines the Fractal Market Hypothesis (FMH) by proposing the following five underlying assumptions:

- FMH 1: The market is made up of many individuals with a large number of different investment horizons.
- FMH 2: The information has a different impact on different investment horizons.
- FMH 3: The stability of the market is mostly a matter of liquidity (balancing of supply and demand). Liquidity is available when the market is composed of many investors with many different investment horizons.
- FMH 4: Prices reflect a combination of short-term technical trading and long-term fundamental valuation.
- FMH 5: If security has no tie to the economic cycle, then there will be no long-term trend. Trading, liquidity, and short-term information will dominate.

The FMH, on the contrary of the EMH, is based on chaos theory and emphasises the impact of information and investment horizons on the behaviour of investors (Rachev et al., 1999; Weron, 2000).

Most studies focus on developed markets. Moreover, what about emerging markets or equivalents as such. Can not they be efficient if they respect international standards? These questions and others related to them motivated our choice in addition to the performance noted in this market. Moreover, its integration into the MSCI Frontier Markets Index is justified by the significant evolution of its leading indicators over the last four years, notably: capitalisation (increased from FCFA 4031 billion at 31 December 2012 to 7500 billion FCFA at 31 December 2015, an increase of 86%); traded volumes (from 41 million shares sold in 2012 to 114 million traded in 2015, a rise of 178%); the annual value of transactions (increased from 146 billion FCFA in 2012 to 336 billion FCFA in 2015, an increase of 130%). This article contributes to the existing literature, analysing the efficiency of the WAEMU Stock Exchange, which is not yet developed. To participate in the visibility of this market for the attraction of investors.

We use new approaches that take into account the limits of the Brownian model. We use the wavelet approach and the multifractal detrended fluctuation analysis. The rest of the document is structured as follows. The literature review of financial market efficiency hypothesis is presented in Section 2. In Section 3, the methodology used in detailed. In Section 4, we discuss the results, and finally, in Section 5 we present the conclusion.

2 Literature Review

As previously stated, the financial markets efficiency theory has attracted much interest from the academic community; various tests categories were used. In a weak sense, the predictability tests are generally used. Hence the utilisation of "random walk" is possible, an assumption that an analysis of past (current)prices cannot allow a present forecast price (future). The hypothesis confirmation may be favourable to efficiency in the weak form.

First-order linear autocorrelation tests (that is, between t and $t - 1$) very often led to results consistent with the random walk hypothesis (Fama, 1965; Working, 1934; Kendall & Debreu, 1953; Osborne, 1959; Alexander, 1961; Samuelson, 1965; Hagin, 1966; Niederhoffer & Osborne, 1966; Sharma & Kennedy, 1977; LeRoy, 1973; Lucas, 1978; Bondt & Thaler, 1985). While another group of authors rejects this hypothesis (Cootner, 1964; Cowles & Jones, 1937). They argue that stock market price variations have some dependence, which has led to the publication of several books, including Cootner (1964).

Other tests in the literature can test this hypothesis (the unit root tests, the variance ratio test, the run test, and the BDS test, etc.) (Chowdhury, 1994; Choudhry, 1994; French & Roll, 1986; Summers, 1986; Poterba & Summers, 1988; Lo & MacKinlay, 1988; Bondt & Thaler, 1985; Urrutia, 1995; Barnes, 1986; Worthington & Higgs, 2006; Sharma & Kennedy, 1977; Seiler & Rom, 1997; Ryoo & Smith, 2002; Chang & Ting, 2010; Kim et al., 1991; Borges, 2007; Kwiatkowski et al., 1992; Phillips & Perron, 1988). The studies on the co-integration relationship are Hakkio & Rush (1989); Mobarek & Fiorante (2014). However, there is a new trend, including the utilisation of wavelets and Multifractal Fluctuation Analysis (MF-DFA). These two approaches make it possible to take into account the share prices behaviour at different time scales.

Instead of the traditional methods, Kumar & Kamaiah (2014) uses the wavelet method to study the weak form efficiency of the NASDAQ, DJIA and S&P indices (from 04-01-1980 to 12-09-2013). They used multi-scale entropy analysis by a MODWT decomposition and extracted sample entropy measure across different timescale. They find that markets are informationally efficient in a weak sense only in the long term (semi-annual, annual). That is to say; as the time horizon increases, the markets evolve towards efficiency. Simonsen et al. (1998) proposes a multi-scale method, the Average Wavelet Coefficient Method (AWC) to compute the Hurst Exponent. The AWC method is a multi-scale method, in the sense where the behaviour at different scales does not influence each other in any significant way, i.e., the technique decouples scales. Fernández-Martínez et al. (2016) proposes a new method based on a multi-scale lifting to estimate the Hurst exponent. The advantages of this approach to the existing Hurst parameter estimation is that it naturally copes with data sampling irregularity. They show that virtually all current Hurst parameter estimation methods which assume a regularly sampled time series and require modification to deal with variability or missing data which introduce higher estimator bias and variation.

Many previous studies make use of the Hurst exponent in the analyses of weak form market efficiency (Pascoal & Monteiro, 2014; Kumar & Kamaiah, 2014). Wang et al. (2009) using MF-DFA divides their series into sub-series and finds that Shenzhen stock market was becoming more and more efficient by analysing the change of Hurst exponent and a new practical measure, which is equal to multifractality degree sometimes. They also showed that the volatility of the series still have significantly long-range dependence (LRD) and multifractality indicating that some conventional models such as GARCH and EGARCH cannot be used to forecast the volatilities of Shenzhen stock market. Lahmiri (2017) studies the multifractality of Moroccan family business stock returns. The results show that short (long) fluctuations in family business stock returns are less (more) persistent (anti-persistent) than small fluctuations in market indices. Also, both serial correlation and distribution characteristics significantly influence the strength of the multifractal spectrum of CSE and family business stocks returns. Furthermore, results from the multifractal spectrum analysis suggest that family business stocks are less risky. Thus, such differences in price dynamics could be exploited by investors and forecasters in active portfolio management.

Ikeda (2018), using the MF-DFA, shows that a multifractal structure characterises the Russian stock market. The author concludes that the multifractality degree of the Russian stock market can be categorized within emerging markets. Suárez-García & Gómez-Ullate (2014) shows that the high-frequency returns of the Madrid's Stock Exchange Ibex35 index exhibit a broad singularity spectrum which is most likely caused by its long-memory. Tiwari et al. (2017), using Hurst exponent and multifractal detrended fluctuation analysis (MF-DFA) methods, shows the multifractality of this index. Other essential results are the utilities and consumer goods sector ETF markets are more efficient compared with the financial and telecommunications sector ETF markets, in terms of price prediction, , there are noteworthy discrepancies in terms of market efficiency, between the short- and long-term horizons and the ETF market efficiency is considerably diminished after the global financial crisis. Khamis et al. (2018) applies the MF-DFA approach to study the efficiency of the Bitcoin market compared to gold, stock and foreign exchange markets. They found that the long-memory feature and multifractality of the Bitcoin market was stronger and therefore, more inefficient than the gold, stock and currency markets.

3 Methodology

In this section, we briefly present the methods of analysis used in the weak efficiency analysis: the wavelet-based methods and the multifractal detrended fluctuation analysis method. The advantage of these approaches is that the wavelet method can eliminate some trend as a result of vanishing moment property and the MF-DFA method allows to avoid spurious detection of correlation that are artefacts of the non-stationarity stock market index.

3.1 Wavelet-based

3.1.1 Wavelet-based Hurst Estimation

Many authors have proposed the wavelet method for estimating the Hurst exponent (Abry & Veitch, 1998) and the method has been improved by Simonsen et al. (1998); Abry et al. (2000); Fernández-Martínez et al. (2016); Abry & Véhel (2013). The advantage of this approach is that it permits to capture the time-varying proprieties of Long memory process and the self-similarity calling behaviour. Abry & Veitch (1998) shows that the Hurst coefficient estimator obtained by the wavelet method is unbiased and efficient under certain general conditions.

Let be $X(t)t \in [0, n]$ a stationary series of second order spectrum Γ_ν that satisfies the following conditions

$$\Gamma_X(\nu) \sim C_X(\nu)^{-(2H-1)}; |\nu| \longrightarrow 0 \text{ and } \frac{1}{2} < H < 1 \tag{1}$$

Let be $\psi(t)$ the mother wavelet, and the wavelet coefficients are obtained by:

$$d_X(j,k) = \int_{\mathbb{R}} \psi_{j,k}(t)x(t)dt \quad \text{or} \quad \psi_{j,k}(t) = 2^{-j/2}\psi(2^{-j}t - k) \tag{2}$$

If $X(t)$ is stationary,

$$E_{d_X}(j,k) = \int_{\mathbb{R}} |\Gamma_X(\nu)\Gamma(2^j\nu|d\nu \tag{3}$$

Taking the density function property, the equation (3) becomes

$$E_{d_X}(j,k) \simeq C_X 2^{j(2H-1)}\ 2^j \longrightarrow \infty \tag{4}$$

By taking the log of equation (4) we find

$$\ln(E_{d_X}(j,k)) \sim j(2H-1) + \ln_2 C_X \quad \text{with } 2^j \longrightarrow \infty \tag{5}$$

By asking

$$S_j = \frac{1}{n_j}\sum_{k=1}^{n_j} d_X(j,k)^2 \tag{6}$$

where n_j is the number of wavelet coefficients available at the 2^j scale, the estimate of $E(d_X((j,k))^2$ can be done using S_j by a weighted linear regression because of heteroscedasticity of $\log_2 S_j$.

$$\ln_2(S_j) = (\ln_2 e)^2 C(j) \tag{7}$$

3.1.2 Wavelet-Based Multiscal Diagram

The procedures for estimating the Hurst exponent (H) have undergone several extensions by following Abry & Veitch (1998). We can thus have a H which depends on q; $H(q)$, the case of multifractality. This method studies the variation of H using the multifractal spectrum of Legendre. The partition function can obtain this spectrum.

$$Sq(\tau) = \int |X(\omega, t+\tau) - X(\omega,t)|^q dt \tag{8}$$

with

$$\tau \longrightarrow 0\ |X(\omega,t+\tau) - X(\omega,t)| = |\tau|^{H(\omega,t)} \tag{9}$$

The transformation of the Legendre function $\xi(q)$ by the Legendre function gives the multifractal spectrum of Legendre. Abry & Véhel (2013) uses multi-scale diagrams to study the multifractal process. The wavelet processes are an implement adapted to the calculation of $\zeta(q)$

$$\int \mid \Gamma_X(a,t) \mid^q dt = a^{\zeta(q)+1/2q} \quad \text{with } a \longrightarrow 0 \tag{10}$$

$$\text{where } T_X(a,t) = \int \frac{1}{\sqrt{a}} X(\omega)\psi\left(\frac{\mu - t}{a}\right) d\mu \tag{11}$$

According to the law-power

$$S(j,q) = \frac{1}{n_j}\sum_{k=1}^{n_j} \mid d_X(j,k) \mid^q \simeq 2^{j\zeta(q)} \in \mathbb{R} \text{ for small j} \tag{12}$$

We can thus estimate $\zeta(q)$ by linear regression. If $\zeta(q) = hq$, the obtained process is monofractal, the spectrum is determined by H is a self-similar process and LRD. However, if $\zeta(q)$ is nonlinear we have a multifractal process.

3.2 Wavelet Leader for Multifractal Analysis

We describe in this subsection the "wavelet leader for multifractal" procedure (Wendt et al., 2009). Let $\psi(t)$ be a compact time support mother wavelet function and N vanish moments. Let $\lambda_{j,k} = \left[k2^j \ (k+1)2^j\right[$ a dyadic interval and $3\lambda_{j,k}$ the interval defined by

$$3\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1} \tag{13}$$

The wavelet Leader is defined as the local supremum of the wavelet coefficients taken within a spatial neighborhood overall more excellent scale (Jaffard & Mélot, 2005; Wendt et al., 2007):

$$L_x(j,k) = \sup_{\lambda' \cup 3\lambda} |d'_\lambda| \tag{14}$$

$L_x(j,k)$ consists of the most significant wavelet coefficient $d_x(j'k')$ calculated at all finer scales $2^{j'} \geq 2^j$ within a narrow time neighborhood $(k-1)2^j \geq 2^j < (k+2)2^j$

3.2.1 Wavelet-Leader Multifractality Formalism

The wavelet leader multifractal formalism allows to estimate D(h) from the defined structure function:

$$S(q,j) = \frac{1}{n_j}\sum_{k=1}^{n_j} L_x^q\,(j,k) \tag{15}$$

where n_j is the number of Leaders available at scale 2^j is q-moment of $L_X(j,k)$ at scale. Assuming structural functions Behave at Scale 2^j

$$S\,(q,j) \simeq C^L(q)2^{j\zeta(q)} \tag{16}$$

A Legendre transformation of scale exponent $\zeta(q)$ which allows an estimation of the multifractal spectrum (Jaffard & Mélot, 2005);

$$\mathfrak{L}(h) = inf_q\,(1 + qh - \zeta(q)) \leq D(h) \tag{17}$$

where

$$\zeta(q) \triangleq \lim_{j\to\infty} inf \frac{\log_2 S(q,j)}{-j} \tag{18}$$

$S(q,j) = 2^{-j\zeta(q)}\ j \longrightarrow +\infty$. Also, the structure-function can be read as an estimate of the mean for the whole averages $\mathbb{E}(L_x(j,k))^q$ so that the exponent of scale is a function of log-cumulants.

3.2.2 Log-cumulants

To solve the difficulties of estimating the $\zeta(q)$ function for all q; Wendt et al. (2007) proposes to use a polynomial in the regression mode (19).

$$\zeta(q) = \sum_{p\geq 1} cp\frac{q^p}{p!} \tag{19}$$

where the coefficients c_p can be related to the cumulants of p order and satisfy $\forall p \geq 1$:

$$c_p(j) = c_{0p} + c_p \ln 2^j \quad \forall p \geq j \tag{20}$$

with the constant c_{0p} does not play any role in the fractal analysis.

We note that:

i) Using only the first two cumulants, we arrive at a good approximation of $\zeta(q)$.
$\zeta(q) \simeq c_1(q) + \frac{1}{2}q^2$ et $D(h) \simeq 1 + \frac{(h-c_1)^2}{2c_2}$

ii) The definition of cumulants implies that for $p^{'} > p$ if $c_p = 0$ so $c_{p'} = 0$ (for more detail see Jaffard et al. (2006) and Wendt et al. (2009)).

The meanings of the log-cumulants possess a certain specificity: c_1 mainly characterises the location of the maximum of D (h), c_2 corresponding to its width and c_3 corresponding to its asymmetry. Thus, all multifractal information of the signal X is contained in the triplet $(c_1; c_2; c_3)$.

Kantelhardt et al. (2002) shows under the specific condition of uniformity of Hölder

$$S_q^L(j) \sim Fq2^{j\zeta(q)} \quad \text{with } 2^j \longrightarrow 0 \tag{21}$$

The estimator $\hat{\zeta}(q)$ of $\zeta(q)$ is obtained by a linear regression of j on $\ln_2 S_q^j$

$\hat{\tau}(q) = \sum_{j=j_1}^{j_2} \omega_j \ln_2 S_q^L(j)$.

The ponderation coefficients satisfy: $\sum_{j=j_1}^{j_2} j\omega_j \equiv 1$ et $\sum_{j_1}^{j_2} \omega_j \equiv 0$

Wendt et al. (2012) proposes a parametric procedure to estimate the multifractal spectrum that writes the Legendre transform

$$\hat{f}(q) = \sum_{j=j_1}^{j_2} \omega_j \cup^L (j,q)$$
$$\hat{h}(q) = \sum_{j=j_1}^{j_2} \omega_j \vee^L (j,q)$$

where

$$\cup^L(j,q) = \sum_{k=1}^{n_j} R_X^q(j,k) \ln_2(j,k) + \ln_2 n_j$$
$$R_X^q = L_X(j,k)^q / \sum_{k=1}^{n_j} (j,k)^q$$

The wavelet transform eliminates polynomial trend of order $N-1$.

3.3 Multifractal Detrended Fluctuation Analysis

We followed the MF-DFA procedure suggested by Kantelhardt et al. (2002). The process consists of five steps in which the first three steps are analogous to the conventional DFA procedure introduced by Peng et al. (1994). Suppose that x_k represents a time series of finite length N with an insignificant fraction of zero values and if there is any zero value exist, i.e. $x_k = 0$, it will be interpreted as having no value at k.

Step 1: We determine the profile

$$Y(i) = \sum_{k=1}^{i} [x_k - \langle x \rangle] \tag{22}$$

where $\langle x \rangle$ denotes the mean of the entire time series.

Step 2: We divide the profile Y(i) into $\lfloor \frac{N}{s} \rfloor$ non-overlapping segments of equal length s. Since the length N may not always be the multiple of s where some end part of the profile may remain, the same procedure is repeated starting from the opposite end of the profile so that the remaining data is not ignored. As a result, we obtained a total of 2Ns segments altogether.

Step 3: We compute the local trend of each of the 2Ns segments using the least-square fit of the series. After that, we compute the variance of each ν^{th} segment. For each segment $\nu = 1, \ldots, Ns$, the variance can be obtained by:

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} \left\{ Y \left[(\nu - 1)s + i \right] - y_\nu(i) \right\}^2 \tag{23}$$

and for each segment, $\nu = 1, \ldots, N$, the variance can be found by:

$$F^2(\nu, s) = \frac{1}{s} \sum_{i=1}^{s} \left\{ Y \left[N - (\nu - N_s)s + i \right] - y_\nu(i) \right\}^2 \tag{24}$$

where $y_\nu(i)$ is the fitting polynomial i.e. the local trend in the ν^{th} segment. Linear (MF-DFA1), quadratic (MF-DFA2), cubic (MF-DFA3), or higher order polynomials can be used in the fitting procedure. Since the detrending of a time series is done by subtracting the fits from the profile, different degrees of polynomial differ in their capability of eliminating trends in the series. Thus, one can estimate the type of polynomial trend in a time series by comparing the results for different detrending orders of MF-DFA (e.g., Peng et al. (1994); Wang et al. (2009); Kantelhardt et al. (2002)).

Step 4: Calculate the qth order fluctuation function averaging all segments $\nu = 1, \ldots, 2N_s$:

$$F_q(s) = \left\{ \frac{1}{2N_s} \sum_{\nu=1}^{2N_s} \left[F^2(\nu, s) \right]^{q/2} \right\}^{1/q} \tag{25}$$

where $q \in \mathcal{R}$. As q approaches zero, the averaging procedure in (25) cannot be applied directly because of the diverging exponent. Therefore, the following logarithmic averaging method is employed as a substitute for $q = 0$:

$$F_0(s) = \exp\left\{\frac{1}{4N_s}\sum_{\nu=1}^{2N_s}\ln\left[F^2(\nu,s)\right] \sim s^{h(0)}\right\} \tag{26}$$

Note that $h(0)$ cannot be defined for times series with fractal support where $h(q)$ diverge for $q \to 0$. The step 2 to step 5 are repeated for numerous time scale s. It is obvious that as s increases, the value of Fq(s) will increase. Note that $h(0)$ cannot be defined for times series with fractal support where $h(q)$ diverge for $q \to 0$. Step 2 to step 5 are repeated for various time scale s. It is evident that as s increases, the value of Fq(s) will increase.

Step 5: We analyse the slope of log-log plots of $Fq(s)$ versus s for each value of q to determine the scaling behaviour of fluctuation functions. The value of $Fq(s)$ will increase as a power-law for large value of s if the series x_k are long-range power-law correlated:

$$F_q(s) =\sim s^{H(q)} \tag{27}$$

Remark 3.1 *For very high value of s, $s > N/4$, $F_q(s)$ becomes imprecise due to estimation errors for small segments of size N_s. For better precision we choose $s < N/4$ with minimum value $s = 2^5$. We will often take in practice $s < 30$ to eliminate spurious results $h(q)$ can be graphically analysed by log-log-plot of $F_q(s)$ depending of s.*

The scaling exponent $H(q)$ in (27) generally may depends on q. However, in a monofractal time series, $H(q)$ is independent of q since the scaling behavior of the variances in (23) and (24) is identical for all segments ν. On the contrary, in a multifractal time series, there will be a notable dependence of $H(q)$ on q due to the different scaling behaviours in the small and large fluctuations. The scaling exponent $H(q)$ is known as the generalised Hurst exponent seeing that $H(2)$ is identical to the well-known Hurst exponent. The relationship between the generalised Hurst exponent $H(q)$ and the classical multifractal scaling exponent $\tau(q)$ can be established using:

$$\tau(q) = qH(q) - 1 \tag{28}$$

We can determine the degree of similarity (Schumann & Kantelhardt, 2011), (or strength of multifractality) infinite limit [–q, +q] by:

$$\Delta H(q) = H(qmin) - H(qmax). \tag{29}$$

We can use the singularity spectrum $f(\alpha)$ to describe the multifractal data and the α parameter is the hölder exponent. We can use the (29) via the Legendre transform to establish the equations of the singularity spectrum and the $f(\alpha)$ parameter is the hölder equation (30):

$$\alpha = H(q) + qH'(q) \quad \text{and } f(\alpha) = q[\alpha - H(q)] + 1 \tag{30}$$

Another quantify for the multifractality degree for the same limit is:

$$\Delta\alpha = |\alpha_{qmin} - \alpha_{qmax}| \tag{31}$$

4 Data and Empirical Results

4.1 Data

We apply the proposed methodology to the daily series of BRVM10 Index from 11 April 2008 to 23 August 2016.[2] All the tables and figures are prepared by us from the data of the WAEMU. The choice of the BRVM10 index is explained by the fact that we want to work with the most liquid assets of the WAEMU for studying the efficiency. The data is transformed into a series of daily price return r_t. The stock market returns were calculated as follows.

$$r_t = \ln\left(P_t/P_{t-1}\right) \tag{32}$$

where P_t and P_{t-1} are the closing price of an index on t and t-1 respectively.

It represents the continual evolution at the level of BRVM10 index of WAEMU stock market, Figure 1-a. We note periods of decline and rise that can be explained by the periods of crises noted in Ivory Coast. Figure 1-b on the right shows the presence of clustering effect which is characterised by a grouping of extremes.

The descriptive statistics are presented in Table 1 below. The log-returns average is positive. The returns series is asymmetric and leptokurtic according to asymmetry coefficients> 0 and kurtosis> 3. These results are in line with the Jarque-Bera and Kolmogorov normality tests, which reject both the normality assumption of the distribution (P-value $0.0000 < 5\%$).

Table 1: Descriptive Statistics of rt

obs	mean	min	max	st-dev	skwness	kurtosis	jarque-Bera	Kolmogorov
2,048	0.0003	-0.1210	0.1462	0.0110	0.2836	25.7891	610 (0.0000)	0.13042 (0.0000)

4.2 Empirical Results and Discussion

In Table 2, $0.5 > H > 1$ indicates that the process increments are positively correlated and that the process exhibits a long-term dependency and a persistent fractional Brownian motion. Our results suggest that price increases are positively associated with each other (Joseph effect). Positive or negative trends are likely to be pursued in the same direction. Moreover, the fractal dimension that is given by $1/H$ corresponds in our case to a size less than 2. In other words, links of dependency connect stock prices over time. The results indicate that small fluctuations have a higher $H(q)$ than significant volatility. The WAEMU stock market has long memory characteristics.

[2] The BRVM10 Index is composed of the ten most active companies in the WAEMU regional stock exchange according to different criteria (market capitalisation, trading volume per trading session and frequency of transactions). The BRVM 10 is revised four times a year (the first Monday in January, April, July and October)

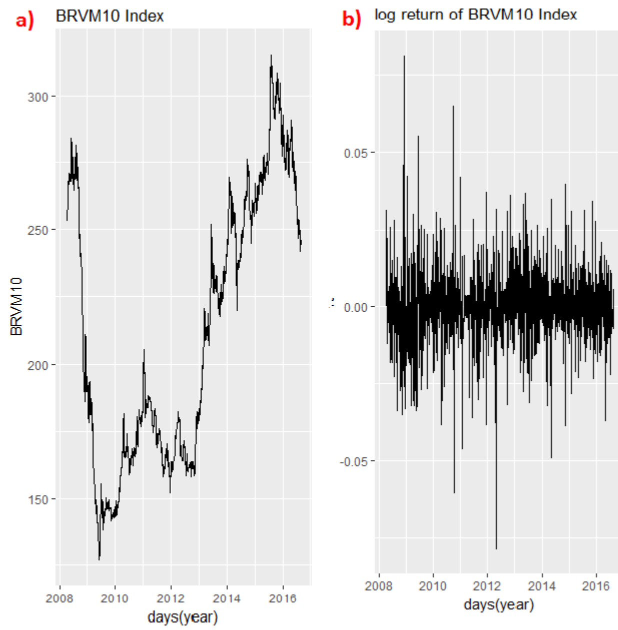

Figure 1: Dynamics of BRVM10 Index and Its log Return

Table 2: Wavelet Hurst exponent estimation results

Coefficient	awc Hurst	FEXPMS	LD estimate	WaveBBJadaptif	genhurstw
Hurst-exponent	0.6304	0.5449	0.578	0.5440	0.5767

Table 3 shows the evolution of scale exponent, multifractal density and generalised Hurst index. The estimators of these different variables are obtained using the Discrete Wavelet Transformation (DWT) and the Leaders Wavelet Transformation (LWT) method.[3] Note that the generalised index $H(q)$ varies in function of q, it decreases as q increases. In other words, the logarithmic return exhibits a multifractal process.

In Figure 2, we can see that $\zeta(h)$ deviates from the linear behaviour of a monofractal process. In both cases, $\zeta(h)$ shows a downward concavity characteristic of the multifractal processes. Equivalently, $D(h)$ shows significant support that is not reduced to a single point, as indicated by the confidence intervals of the last points which, although broad, do not overlap. These results suggest that our analysis should instead be based on the multifractal paradigm.

In Figure 3-b at the top and right, the value of $H(q)$ varies according to q. Like the wavelet method, the generalised Hurst exponent decreases as q increases. On the bottom right, Figure 3-d, we have the multifractal spectrum, and the maximum is close to 1, we have $0.5 < H < 1$, so we have a multifractal process.

[3] For the sake of space the table has been reduced to the interval [-5,5], but we have worked on the range [-10,10].

Table 3: Hurst estimation by DWT and LWT

Moments	ESTIMATES DWT			ESTIMATES LWT		
q	$\zeta(q)$	**D(q)**	**H(q)**	$\zeta(q)$	**D(q)**	**H(q)**
-5	-6.124*** (1.852)	-0.212 (0.336)	1.467*** (0.387)	2.833*** (0.222)	0.135 (0.181)	0.74*** (0,071)
-4	-4.661*** (1.468)	-0.171 (0.336)	1.458*** (0.396)	2.123** * (0.160)	0.504*** (0.148)	0,678*** (0.064)
-3	-3.212*** (1.075)	-0.104 (0.340)	1.439*** (0.415)	1.480** * (0.107)	0.667*** (0.074)	0.604*** (0.053)
-2	-1.790*** (0.656)	-0.002 (0.374)	1.396*** (0.466)	0.915 *** (0.064)	0.861*** (0.031)	0.527*** (0.041)
-1	-0.534*** (0.206)	0.585 (0.239)	0.949*** (0.424)	0.424*** (0.029)	0.968*** (0.007)	0.456*** (0.032)
1	0.261*** (0.071)	0.970*** (0.028)	0.231*** (0.075)	0.362*** (0.029)	0.969*** (0.010)	0.332*** (0.034)
2	0.457*** (0.148)	0.853*** (0.087)	0.155* (0.094)	0.662*** (0.068)	0.871*** (0.043)	0.266*** (0.05)
3	0.569** (0.238)	0.649*** (0.140)	0.073 (0.104)	0.895*** (0.124)	0.705*** (0.096)	0.2*** (0.068)
4	0.612* (0.333)	0.467*** (0.169)	0.02 (0.101)	1.065*** (0.194)	0.504*** (0.148)	0.142* (0.079)
5	0.618 (0.423)	0.358 (0.182)	0.005 (0.094)	1.184*** (0.271)	0.314 * (0.187)	0.1* (0.084)

Notes: Standard errors are in parentheses, * 10% significant, ** 5% significant, *** 1% significant

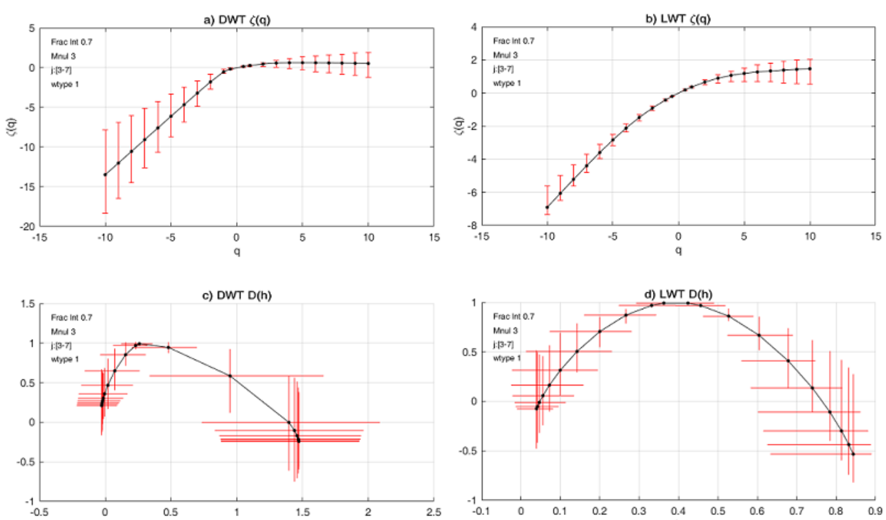

Figure 2: Scaling Exponents and Multifractal Spectrum

Note: The Scaling exponents $\zeta(q)$ DWT top row (left column) and $\zeta(q)$ DWT top row (right column) $D(h)$ DWT (left top row (left column)) and) $D(h)$ LWT (bottom row (right column). Red lines indicate bootstrap-based 95% confidence.

Figure 4 shows $\zeta(h)$ and $D(h)$ log-return obtained with DWT and LWT. The red lines indicate the 95% confidence intervals based on the bootstrap (Wendt et al., 2007). We have a structure-function for q (-2, -1, 1, 2). We can see that ζ deviates from the linear behavior of an “equivalent” monofractal process that is, with $H = c1$, illustrated with blue dashed

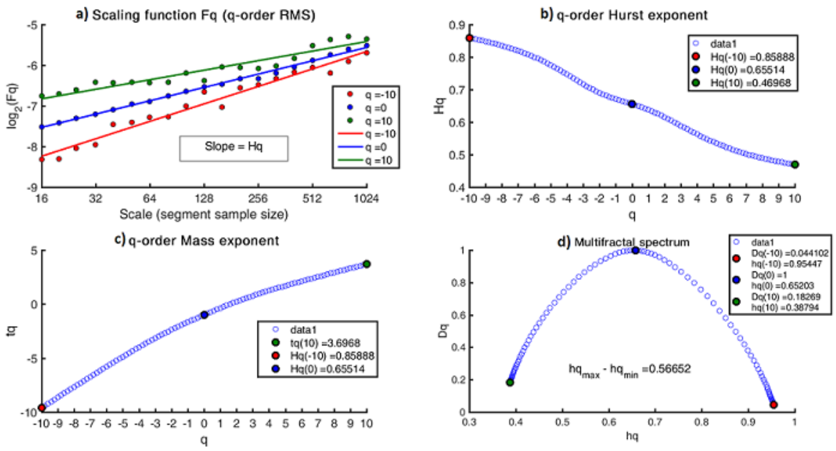

Figure 3: Multifractal analysis by MF-DFA

lines. Blue crosses show the spectra for the equivalent monofractal process. The differences between the "shaped" estimated spectra and the monofractal one are evident.

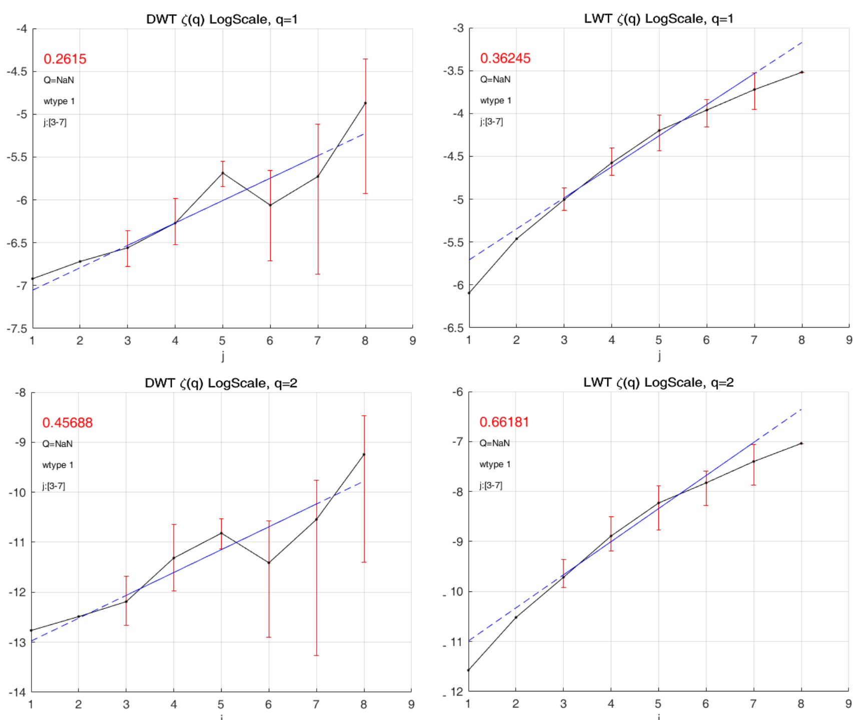

Figure 4: Structure Functions

Note: $log_2 S(q,j)$,DWT (left top row) and $log_2 S(q,j)$, LWT (right top row). $log_2 S(q,j)$, DWT (left bottom row) for q=1 and $log_2 S(q,j)$, LWT (right bottom row), for q=é, (black lines), bootstrap-based 95% confidence intervals (red lines) and least square linear fit (blue line).

Figure 5 c_1 is significant regardless of the method used. c_2 is meaningful only with the LWT method. On the other hand, c_3 not significant whatever the technique. The results of c_1 and c_2 confirm that the logarithmic return follows a multifractal process.

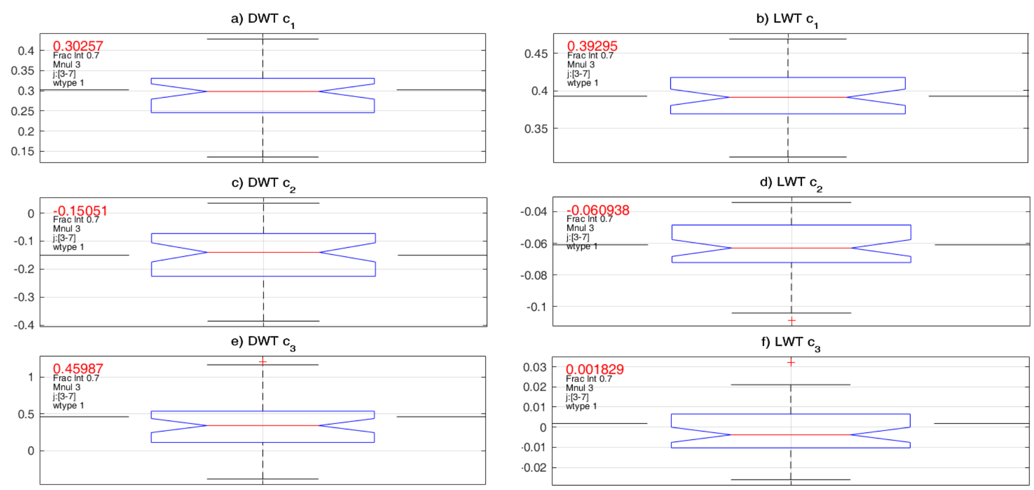

Figure 5: Log-cumulants Estimations

Note: Boxplots of the estimations of c1 DWT top row (left column) and c1 LWT (right column), c2 DWT middle row (left column) and c2 LWT (right column) C3 DWT bottom row (left column) and c2 LWT (right column). Red '+' signs indicate points that where considered as extremes.

- The first cumulant is the slope estimate; in other words, it captures the linear behaviour.
- The second cumulant captures the first departure from linearity. A reader can think of the second cumulant as the coefficients of a second-order (quadratic) term,
- while the third cumulant characterises a more complicated departure from the exponents of the scaling of the linearity

Table 4: Log cumulant estimate

Cumulant	Estimate DWT	Estimate LWT
c_1	0.303** (0.081)	0.393** * (0.028)
c_2	-0.151 (0.152)	0.061 ** (0.016)
c_3	0.460 (0.497)	0.002 (0.010)

Notes: Standard errors are in parentheses, * 10% significant, ** 5% significant, *** 1% significant

In Figure 6, we observe an apparent linear behaviour for the different scales, especially the graphs obtained from LWT. The blue lines represent the linear adjustments obtained in each case, showing a perfect fit with the data for the different scales selected and confirms that the data verify the theoretical invariance properties of scale models. A general requirement on multifractal models is that the behaviour of scales where the scale invariance observed must be the same for all statistical orders, i. e., q in the case of S (j, q); or p in the case of $c_p(j)$. This is indeed the case in the figure, where the scales associated with each j belongs [3: 7]. The appropriate graphs are observed for both $S(j,q)$ and and for c_p. (1), for all statistical orders and both approaches. This last remark could have an interesting financial interpretation, as it seems to suggest that the financial mechanisms responsible for scale invariance operate at the same time.

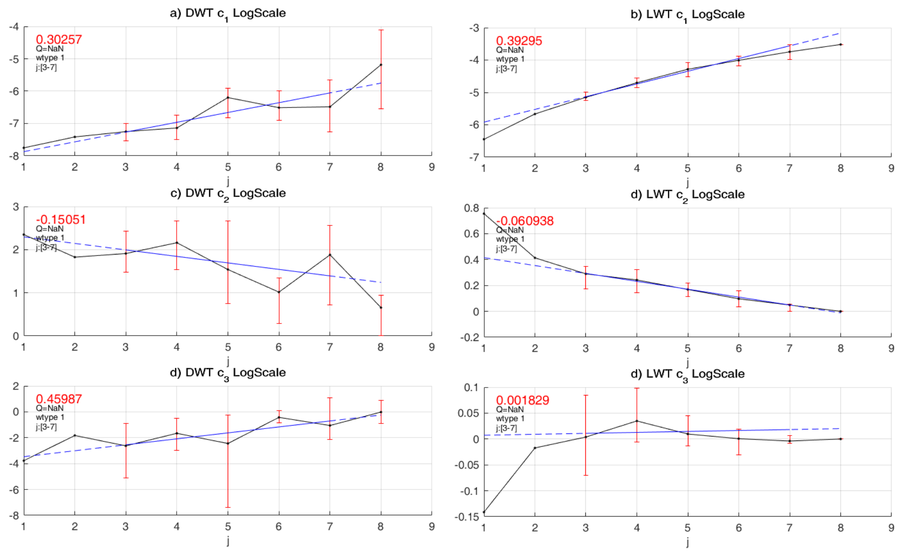

Figure 6: Log Cumulants

Note: Log Cumulants (top row) and log2(e) Cp(j) middle row, (bottom row), DWT, (left column) and LWT (right column), for several q and p (black lines), bootstrap-based 95% confidence intervals (red lines) and least square linear fit (blue line)

5 Conclusion and Policy Implications

This study allowed us to examine the efficient market hypothesis of the financial markets through the BRVM10 index of the local stock exchange, which groups together the 8 WAEMU countries. We used the wavelet approach and multifractal detrended fluctuations analysis. The series studied show tails of distribution thicker than those of a normal distribution. Our results indicate that the WAEMU Regional Stock Exchange is multifractal. We have a Hurst $1/2 < H < 1$, so we have a persistent process. The BRVM10 index conforms to the multifractality principle of the financial markets. However, it is essential to note that small fluctuations have higher persistence dynamics. As previously mentioned, parasitic multifractality can be induced by complex non-stationarities, which often exist in the studied data, for example in the hydrological domain because of the seasonal cycle or a change of climate. In our study, multifractality is confirmed by several methods (multi-scale diagram, MF-DFA, and WLMF). The financial market efficiency can provide a global view of the situation of a market. It would make investments attractive because of the ability of this market to direct funds towards the most productive jobs and a lack of insider trading. The WAEMU stock exchange performed strongly, with the BRVM10 and BRVM Composite indices respectively 94.61 and 98.04 points at the introduction of the index, compared to 219.65 and 243.06 points respectively at 31 December 2017, representing increases of 132% and 148%, respectively. In addition, the WAEMU has been integrated into the MSCI and S&P indices (www.brvm.org). But it has experienced periods of decrease of 46% because of the fall of cocoa (between January 2016 and December 2017) following a disappointing agricultural campaign (www.brvm.org; 02/09/18). Besides, we have an informal sector which occupies a considerable weight in our countries and the lack of policy of accompaniment of the private companies. Also, fears about the stability and sustainability of the CFA franc that revive the debate is a brake on investment. Thus, policy-makers should review whether the stock exchange respect international standards globally to put in place a reliable regulatory system. Review the level of liquidity to be able to cope with other stock markets both on products and the level of technology used.[4]

The institutional, legal and regulatory framework can impact on the functioning of the market (Merton, 1992; Jayasuriya, 2005). The member countries of WAEMU could even be forced to work on a possible relocation of the head office to a much more stable member country to eliminate any risk (however small) of curbing investors because of the recurring Ivorian crises. Also, a relaxation of regulations and stock exchange procedures to attract more partner could be beneficial. It could be encouraged other member countries to be more involved with the exchange; the Ivory Coast alone holds 35 of the 45 companies listed (www.brvm.org; 17/01/2018). In our future studies, we want to broaden this work by conducting a sector-by-sector study of the sub-regional stock market to see which sectors are the most promising to determine the inefficiencies of this stock market. Also, we intend to make a comparative study of African emerging markets.

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[4] "The quality of financial markets can be related to efficiency concept, which is the function of what is called the market microstructure (liquidity, transaction costs, quotation system, etc.), the organisational functioning and institutional (regulation, legal and regulatory framework, transparency, etc.)" (Benjamin, 2007).

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